

Error Estimation for Moments Analysis in Heavy Ion Collision Experiment

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Abstract. Higher moments of conserved quantities are predicted to be sensitive to the correlation length and connected to the thermodynamic susceptibility. Thus, higher moments of net-baryon, net-charge and net-strangeness have been extensively studied theoretically and experimentally to explore phase structure and bulk properties of QCD matters created in heavy ion collision experiment. As the higher moments analysis is statistics hungry study, the error estimation is crucial to extract physics information from the limited experimental data. In this paper, we will derive the limit distributions and error formula based on Delta theorem in statistics for various order moments used in the experimental data analysis. The Monte Carlo simulation is also applied to test the error formula.

1. Introduction

The main goal of performing the higher moments analysis is to study the bulk properties, such as QCD phase transition [1], QCD critical point [2, 3, 4, 5] and thermalization [3], of QCD matters created in the heavy ion collisions experiment. It opens a completely new domain and provides quantitative method for probing the bulk properties of the hot dense nuclear matter. On the other hand, the higher moment analysis can be also used to constrain some fundamental parameters, such as the scale for the QCD phase diagram (the transition temperature T_c at $\mu_B = 0$), by comparing the experimental data with the first principle Lattice QCD calculations [6].

Higher moments (Variance (σ^2), Skewness (S), Kurtosis (κ) *etc.*) of conserved quantities, such as net-baryon, net-charge, and net-strangeness, distributions can be directly connected to the corresponding thermodynamic susceptibilities in Lattice QCD [7, 8] and Hadron Resonance Gas (HRG) model [9], for *e.g.* the third order susceptibility of baryon number ($\chi_B^{(3)}$) is related to the third cumulant ($\langle (\delta N_B)^3 \rangle$) of baryon number distributions as $\chi_B^{(3)} = \langle (\delta N_B)^3 \rangle / VT^3$; V, T are volume and temperature of system respectively. As the volume of the system is hard to determine, the susceptibility ratio, such as $\chi_B^{(4)} / \chi_B^{(2)}$ and $\chi_B^{(3)} / \chi_B^{(2)}$, are used to compare with the experimental data as $\kappa \sigma^2 = \chi_B^{(4)} / \chi_B^{(2)}$ and $S \sigma = \chi_B^{(3)} / \chi_B^{(2)}$. We also measure the ratios of the sixth and eighth to second order cumulants of the net-baryon number fluctuations, as $\chi_B^{(6)} / \chi_B^{(2)}$ and $\chi_B^{(8)} / \chi_B^{(2)}$, respectively, which are predicted to be with negative value when the freeze out temperature is close to the the chiral phase transition temperature. Theoretical calculations demonstrate that the experimental measurable net-proton (proton number minus

anti-proton number) number fluctuations can effectively reflect the fluctuations of the net-baryon number [10]. Thus, it is of great interest to measure the higher moments of event-by-event net-proton multiplicity distributions in the heavy ion collision experiment.

In section 2, we will show the definition of central moments and cumulants. Then, the Delta theorem in statistics will be discussed in section 3 and applied to derive the error formula for various order moments. In section 4, Monte Carlo simulation has been done to check the validity of the error formula. The summary and conclusion will go to the chapter 5.

2. Central Moments and Cumulants of Event-by-Event Fluctuations

In statistics, probability distribution functions can be characterized by the various moments, such as mean (M), variance (σ^2), skewness (S) and kurtosis (κ). Before introducing the above moments used in our analysis, we would like to define central moments and cumulants, which are alternative methods to describe a distribution.

Experimentally, we measure net-proton number event-by-event wise, $N_{p-\bar{p}} = N_p - N_{\bar{p}}$, which is proton number minus antiproton number. In the following, we use N to represent the net-proton number $N_{p-\bar{p}}$ in one event. The average value over whole event ensemble is denoted by $\hat{\mu} = \langle N \rangle$, where the single angle brackets are used to indicate ensemble average of an event-by-event distributions and the hat symbol denotes the sample estimator.

The deviation of N from its mean value are defined by

$$\delta N = N - \langle N \rangle = N - \hat{\mu}. \quad (1)$$

The r^{th} order sample estimates for central moments are defined as:

$$\hat{\mu}_r = \langle (\delta N)^r \rangle \quad (2)$$

$$\hat{\mu}_1 = 0 \quad (3)$$

Then, we can define the sample estimates for various order cumulants of event-by-event distributions as:

$$\hat{C}_1 = \hat{\mu} \quad (4)$$

$$\hat{C}_2 = \hat{\mu}_2 \quad (5)$$

$$\hat{C}_3 = \hat{\mu}_3 \quad (6)$$

$$\hat{C}_n (n > 3) = \hat{\mu}_n - \sum_{m=2}^{n-2} \binom{n-1}{m-1} \hat{C}_m \hat{\mu}_{n-m} \quad (7)$$

An important property of the cumulants is their additivity for independent variables. If X and Y are two independent random variables, then we have $C_{i,X+Y} = C_{i,X} + C_{i,Y}$ for i th order cumulant.

Once we have the definition of cumulants, sample estimators for skewness and kurtosis can be denoted as:

$$\hat{M} = \hat{C}_{1,N}, \hat{\sigma}^2 = \hat{C}_{2,N}, \hat{S} = \frac{\hat{C}_{3,N}}{(\hat{C}_{2,N})^{3/2}}, \hat{\kappa} = \frac{\hat{C}_{4,N}}{(\hat{C}_{2,N})^2} \quad (8)$$

Then, the moments product $\hat{\kappa}\hat{\sigma}^2$ and $\hat{S}\hat{\sigma}$ can be expressed in term of cumulant ratio.

$$\hat{\kappa}\hat{\sigma}^2 = \frac{\hat{C}_{4,N}}{\hat{C}_{2,N}}, \hat{S}\hat{\sigma} = \frac{\hat{C}_{3,N}}{\hat{C}_{2,N}}. \quad (9)$$

With above definition of various moments, we can calculate various moments and moment products with the measured event-by-event net-proton distributions.

3. Delta Theorem in Statistics

Before deriving the limit distributions as well as the error formula of various moments and moment products, we would like to introduce you the delta theorem that used in the calculations. The delta theorem [11, 12, 13] says how to approximate the distribution of a transformation of a statistic in large samples if we can approximate the distribution of the statistic itself. Distributions of transformations of a statistic are of great importance in applications. We will give the theorem with one and multi-dimensional cases without proofs. Before introducing the delta theorem, we will show you an useful theorem of sample moments [12].

Theorem A: If central moments $\mu_{2k} = E[(X - \mu)^{2k}] < \infty$, then the random vector $\sqrt{n}(\hat{\mu}_2 - \mu_2, \dots, \hat{\mu}_k - \mu_k)$ converges in distributions to $(k - 1)$ -variate normal with mean vector $(0, 0, 0, \dots, 0)$ and covariance matrix $[\Sigma_{ij}]_{(k-1) \times (k-1)}$, where

$$\Sigma_{ij} = \mu_{i+j} - \mu_i \mu_j - i \mu_{i-1} \mu_{j+1} - j \mu_{i+1} \mu_{j-1} + ij \mu_{i-1} \mu_{j-1} \mu_2$$

For instance, we have the limit distribution for the sample variance $\mu_2 = \hat{\sigma}^2$:

$$\hat{\sigma}^2 \xrightarrow{d} N(\sigma^2, \frac{\mu_4 - \sigma^4}{n})$$

Then the variance of the sample variance is $Var(\hat{\sigma}^2) = (\mu_4 - \sigma^4)/n$.

In the following, we will introduce the one and multi-dimension delta theorems and their applications.

Delta Theorem-I (One-dimension): Suppose that random variable X distribute as $N(\mu, \frac{\sigma^2}{n})$, let g be a real-valued function differentiable at $x = \mu$, with $g'(\mu) \neq 0$. Then we get the limit distribution of $g(X)$:

$$g(X) \xrightarrow{d} N(g(\mu), [g'(\mu)]^2 \frac{\sigma^2}{n})$$

As an application of the delta theorem, let's estimate the limit distribution of the sample standard deviation $\hat{\sigma}$. It was seen in theorem A that

$$\hat{\sigma}^2 \xrightarrow{d} N(\sigma^2, \frac{\mu_4 - \sigma^4}{n})$$

It follows that the sample standard deviation is also asymptotically normal, namely

$$\hat{\sigma} \xrightarrow{d} N(\sigma, \frac{\mu_4 - \sigma^4}{4\sigma^2 n})$$

, with the function $g(x) = \sqrt{x}$.

The following theorem extends the above delta theorem to the case of a vector-valued function g to a vector $\mathbf{X} = \{X_1, X_2, \dots, X_k\}$.

Delta Theorem-II (Multi-dimension): Suppose that $\mathbf{X} = \{X_1, X_2, \dots, X_k\}$ is normally distributed as $N(\mu, \Sigma/n)$, with Σ a covariance matrix. Let $\mathbf{g}(\mathbf{x}) = (g_1(x), \dots, g_m(x))$, $\mathbf{x} = (x_1, \dots, x_k)$, be a vector-valued function for which each component function $g_i(x)$ is real-valued and has a non-zero differential $g_i(\mu)$, at $\mathbf{x} = \mu$. Put

$$\mathbf{D} = \left[\frac{\partial g_i}{\partial x_j} \Big|_{x=\mu} \right]_{m \times k}$$

Then

$$\mathbf{g}(\mathbf{X}) \xrightarrow{d} N(\mathbf{g}(\mu), \frac{\mathbf{D} \Sigma \mathbf{D}'}{n})$$

In the following sub-sections, we will derive the joint limiting distributions for higher order moments (σ, S, κ) and moment products $(S\sigma, \kappa\sigma^2, \kappa\sigma/S)$. The limit distributions of sixth and eighth to second order cumulants ratio will be also calculated.

3.1. Joint Limiting Distributions of Sample Standard Deviation($\hat{\sigma}$), Skewness (\hat{S}) and Kurtosis($\hat{\kappa}$)

The multi-dimension delta theorem will be applied to derive the joint limit distributions for the sample statistic vector

$$\hat{\mathbf{T}} = \begin{pmatrix} \hat{\sigma} \\ \hat{S} \\ \hat{\kappa} \end{pmatrix}$$

For the sample moments vector

$$\hat{\mathbf{W}} = \begin{pmatrix} \hat{\mu}_2 \\ \hat{\mu}_3 \\ \hat{\mu}_4 \end{pmatrix}$$

We have the limit distributions, when the sample is large enough:

$$\hat{\mathbf{W}} = \begin{pmatrix} \hat{\mu}_2 \\ \hat{\mu}_3 \\ \hat{\mu}_4 \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix}, \frac{\mathbf{\Sigma}}{n}\right)$$

, where the $\mathbf{\Sigma}$ is the 3×3 covariance matrix of the multi-variate vector $\hat{\mathbf{W}}$. The covariance matrix is a symmetrical matrix and the matrix element can be calculated via theorem A.

$$\begin{aligned} \Sigma_{11} &= Var(\hat{\mu}_2) = \mu_4 - \sigma^4 \\ \Sigma_{22} &= Var(\hat{\mu}_3) = \mu_6 - \mu_3^2 - 6\mu_4\sigma^2 + 9\sigma^6 \\ \Sigma_{33} &= Var(\hat{\mu}_4) = \mu_8 - \mu_4^2 - 8\mu_3\mu_5 + 16\mu_3^2\sigma^2 \\ \Sigma_{12} &= \Sigma_{21} = Cov(\hat{\mu}_2, \hat{\mu}_3) = \mu_5 - 4\mu_3\sigma^2 \\ \Sigma_{13} &= \Sigma_{31} = Cov(\hat{\mu}_2, \hat{\mu}_4) = \mu_6 - 4\mu_3^2 - \mu_4\sigma^2 \\ \Sigma_{23} &= \Sigma_{32} = Cov(\hat{\mu}_3, \hat{\mu}_4) = \mu_7 - 3\mu_5\sigma^2 - 5\mu_3\mu_4 + 12\mu_3\sigma^4 \end{aligned}$$

Based on the definition of the Standard deviation, Skewness and Kurtosis, we define a function vector $\mathbf{g} = (g_1 = \sqrt{\mu_2}, g_2 = \mu_3/(\mu_2)^{3/2}, g_3 = \mu_4/(\mu_2)^2 - 3)$. Then we have:

$$\mathbf{D} = \left[\frac{\partial g_i}{\partial \mu_j} \right]_{3 \times 3} = \begin{pmatrix} \frac{\partial g_1}{\partial \mu_2} & \frac{\partial g_1}{\partial \mu_3} & \frac{\partial g_1}{\partial \mu_4} \\ \frac{\partial g_2}{\partial \mu_2} & \frac{\partial g_2}{\partial \mu_3} & \frac{\partial g_2}{\partial \mu_4} \\ \frac{\partial g_3}{\partial \mu_2} & \frac{\partial g_3}{\partial \mu_3} & \frac{\partial g_3}{\partial \mu_4} \end{pmatrix} = \begin{pmatrix} 1/(2\sigma) & 0 & 0 \\ -3\mu_3/(2\sigma^5) & 1/\sigma^3 & 0 \\ -2\mu_4/\sigma^6 & 0 & 1/\sigma^4 \end{pmatrix}$$

Then, according to the multi-variate delta theorem, we have the joint limiting distribution for the random sample vector $\hat{\mathbf{T}}$

$$\hat{\mathbf{T}} \xrightarrow{d} N\left(\begin{pmatrix} \sigma \\ S \\ \kappa \end{pmatrix}, \frac{\mathbf{D}\mathbf{\Sigma}\mathbf{D}'}{n}\right)$$

, where the covariance matrix $\mathbf{\Gamma} = \mathbf{D}\mathbf{\Sigma}\mathbf{D}'/n$ is a 3×3 symmetrical matrix. The matrix element:

$$\begin{aligned} \Gamma_{11} &= Var(\hat{\sigma}) = (m_4 - 1)\sigma^2/(4n) \\ \Gamma_{22} &= Var(\hat{S}) = [9 - 6m_4 + m_3^2(35 + 9m_4)/4 - 3m_3m_5 + m_6]/n \\ \Gamma_{33} &= Var(\hat{\kappa}) = [-m_4^2 + 4m_4^3 + 16m_3^2(1 + m_4) - 8m_3m_5 - 4m_4m_6 + m_8]/n \\ \Gamma_{12} &= \Gamma_{21} = Cov(\hat{\sigma}, \hat{S}) = -[m_3(5 + 3m_4) - 2m_5]\sigma/(4n) \\ \Gamma_{13} &= \Gamma_{31} = Cov(\hat{\sigma}, \hat{\kappa}) = [(-4m_3^2 + m_4 - 2m_4^2 + m_6)\sigma]/(2n) \\ \Gamma_{23} &= \Gamma_{33} = Cov(\hat{S}, \hat{\kappa}) = [6m_3^3 - (3 + 2m_4)m_5 + 3m_3(8 + m_4 + 2m_4^2 - m_6)/2 + m_7]/n \end{aligned}$$

, where the normalized central moments $m_r = \mu_r/\sigma^r$. The non-zero values for the non-diagonal elements of the covariance matrix indicate there are correlation between those three moments

σ, S, κ . When the distribution is a symmetrical distribution, the odd normalized central moments will be zero, thus we have $Cov(\hat{\sigma}, \hat{S}) = Cov(\hat{S}, \hat{\kappa}) = 0$, which means there has no correlation between skewness and the other two moments. For normal distributions,

$$\begin{aligned}\Gamma_{11} &= Var(\hat{\sigma}) = \sigma^2/(2n) \\ \Gamma_{22} &= Var(\hat{S}) = 6/n \\ \Gamma_{33} &= Var(\hat{\kappa}) = 24/n \\ \Gamma_{12} &= \Gamma_{21} = Cov(\hat{\sigma}, \hat{S}) = 0 \\ \Gamma_{13} &= \Gamma_{31} = Cov(\hat{\sigma}, \hat{\kappa}) = 0 \\ \Gamma_{23} &= \Gamma_{32} = Cov(\hat{S}, \hat{\kappa}) = 0\end{aligned}$$

The non-diagonal matrix elements are zero and no correlations between σ, S, κ .

3.2. Joint Limiting Distributions of Sample Moment Products ($\hat{S}\hat{\sigma}, \hat{\kappa}\hat{\sigma}^2, \hat{\kappa}\hat{\sigma}/\hat{S}$)

To derive the joint limiting distributions for the moment products ($\hat{S}\hat{\sigma}, \hat{\kappa}\hat{\sigma}^2, \hat{\kappa}\hat{\sigma}/\hat{S}$), we apply similar procedures as above. Define the random vector:

$$\hat{\mathbf{T}} = \begin{pmatrix} \hat{S}\hat{\sigma} \\ \hat{\kappa}\hat{\sigma}^2 \\ \frac{\hat{\kappa}\hat{\sigma}}{\hat{S}} \end{pmatrix}$$

Based on the definition of the moment products, we define a function vector $\mathbf{g} = (g_1 = \hat{\mu}_3/\hat{\mu}_2, g_2 = \hat{\mu}_4/\hat{\mu}_2 - 3\hat{\mu}_2, g_3 = (\hat{\mu}_4 - 3\hat{\mu}_2^2)/\hat{\mu}_3)$. Then we have:

$$\mathbf{D} = \left[\frac{\partial g_i}{\partial \mu_j} \right]_{3 \times 3} = \begin{pmatrix} \frac{\partial g_1}{\partial \mu_2} & \frac{\partial g_1}{\partial \mu_3} & \frac{\partial g_1}{\partial \mu_4} \\ \frac{\partial g_2}{\partial \mu_2} & \frac{\partial g_2}{\partial \mu_3} & \frac{\partial g_2}{\partial \mu_4} \\ \frac{\partial g_3}{\partial \mu_2} & \frac{\partial g_3}{\partial \mu_3} & \frac{\partial g_3}{\partial \mu_4} \end{pmatrix} = \begin{pmatrix} -\mu_3/\sigma^4 & 1/\sigma^2 & 0 \\ -(\mu_4/\sigma^4 + 3) & 0 & 1/\sigma^2 \\ -6\sigma^2/\mu_3 & -(\mu_4 - 3\sigma^4)/\mu_3^2 & 1/\mu_3 \end{pmatrix}$$

Then, according to the multi-variate delta theorem, we have the joint limiting distribution for the random sample vector $\hat{\mathbf{T}}$

$$\hat{\mathbf{T}} \xrightarrow{d} N\left(\begin{pmatrix} S\sigma \\ \kappa\sigma^2 \\ \kappa\sigma/S \end{pmatrix}, \frac{\mathbf{D}\Sigma\mathbf{D}'}{n}\right)$$

The corresponding matrix element of covariance matrix $\mathbf{\Pi} = \mathbf{D}\Sigma\mathbf{D}'/n$ are

$$\begin{aligned}\Pi_{11} &= Var(\hat{S}\hat{\sigma}) = [9 - 6m_4 + m_3^2(6 + m_4) - 2m_3m_5 + m_6]\sigma^2/n \\ \Pi_{22} &= Var(\hat{\kappa}\hat{\sigma}^2) = [-9 + 6m_4^2 + m_4^3 + 8m_3^2(5 + m_4) - 8m_3m_5 + m_4(9 - 2m_6) - 6m_6 + m_8]\sigma^4/n \\ \Pi_{33} &= Var(\hat{\kappa}\hat{\sigma}/\hat{S}) = [64m_3^4 - 8m_3^3m_5 - (-3 + m_4)^2(-9 + 6m_4 - m_6) + 2m_3(-3 + m_4)(9m_5 - m_7) \\ &\quad + m_3^2(171 - 48m_4 + 8m_4^2 - 12m_6 + m_8)]\sigma^2/(n \times m_3^4) \\ \Pi_{12} &= \Pi_{21} = Cov(\hat{S}\hat{\sigma}, \hat{\kappa}\hat{\sigma}^2) = [4m_3^3 - (6 + m_4)m_5 + m_3(21 + 2m_4 + m_4^2 - m_6) + m_7]\sigma^3/n \\ \Pi_{13} &= \Pi_{31} = Cov(\hat{S}\hat{\sigma}, \hat{\kappa}\hat{\sigma}/\hat{S}) = [4m_3^4 + (-3 + m_4)(-9 + 6m_4 - m_6) - m_3^2(-39 + m_4 + m_6) \\ &\quad + m_3((-12 + m_4)m_5 + m_7)]\sigma^2/(n \times m_3^2) \\ \Pi_{23} &= \Pi_{32} = Cov(\hat{\kappa}\hat{\sigma}^2, \hat{\kappa}\hat{\sigma}/\hat{S}) = [4m_3^3(13 + m_4) - 8m_3^2m_5 + (-3 + m_4)((6 + m_4)m_5 - m_7) \\ &\quad + m_3(54 + 7m_4^2 - 9m_6 - m_4(6 + m_6) + m_8)]\sigma^3/(n \times m_3^2)\end{aligned}$$

, where the normalized central moments $m_r = \mu_r/\sigma^r$. Supposing that the distribution is symmetrically distributed, the variance of the sample statistic $\hat{\kappa}\hat{\sigma}/\hat{S}$ and the corresponding covariance are not well defined. For normal distribution, we have

$$\begin{aligned}\Pi_{11} &= Var(\hat{S}\hat{\sigma}) = 6\sigma^2/n \\ \Pi_{22} &= Var(\hat{\kappa}\hat{\sigma}^2) = 24\sigma^4/n \\ \Pi_{12} &= \Pi_{21} = Cov(\hat{S}\hat{\sigma}, \hat{\kappa}\hat{\sigma}^2) = 0\end{aligned}$$

there has no correlation between sample statistic $\hat{S}\hat{\sigma}$ and $\hat{\kappa}\hat{\sigma}^2$.

3.3. Limit Distribution for Sample Cumulant ratio (\hat{C}_6/\hat{C}_2 and \hat{C}_8/\hat{C}_2)

We only derive the limit distribution for sixth to second cumulant order ratio and give the results for \hat{C}_8/\hat{C}_2 without derivation. For the sample moments vector

$$\hat{\mathbf{W}} = \begin{pmatrix} \hat{\mu}_2 \\ \hat{\mu}_3 \\ \hat{\mu}_4 \\ \hat{\mu}_6 \end{pmatrix}$$

We have the limit distributions, when the sample is large enough:

$$\hat{\mathbf{W}} = \begin{pmatrix} \hat{\mu}_2 \\ \hat{\mu}_3 \\ \hat{\mu}_4 \\ \hat{\mu}_6 \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_6 \end{pmatrix}, \frac{\mathbf{\Omega}}{n}\right)$$

, where the $\mathbf{\Omega}$ is the 4×4 covariance matrix of the multi-variate vector $\hat{\mathbf{W}}$. The covariance matrix is a symmetrical matrix and the matrix element can be calculated via theorem A.

$$\begin{aligned}\Omega_{11} &= Var(\hat{\mu}_2) = \mu_4 - \sigma^4 \\ \Omega_{22} &= Var(\hat{\mu}_3) = \mu_6 - \mu_3^2 - 6\mu_4\sigma^2 + 9\sigma^6 \\ \Omega_{33} &= Var(\hat{\mu}_4) = \mu_8 - \mu_4^2 - 8\mu_3\mu_5 + 16\mu_3^2\sigma^2 \\ \Omega_{44} &= Var(\hat{\mu}_6) = \mu_{12} - 12\mu_5\mu_7 - \mu_6^2 - 36\mu_5^2\sigma^2 \\ \Omega_{12} &= \Omega_{21} = Cov(\hat{\mu}_2, \hat{\mu}_3) = \mu_5 - 4\mu_3\sigma^2 \\ \Omega_{13} &= \Omega_{31} = Cov(\hat{\mu}_2, \hat{\mu}_4) = \mu_6 - 4\mu_3^2 - \mu_4\sigma^2 \\ \Omega_{14} &= \Omega_{41} = Cov(\hat{\mu}_2, \hat{\mu}_6) = \mu_8 - \mu_6\sigma^2 - 6\mu_3\mu_5 \\ \Omega_{23} &= \Omega_{32} = Cov(\hat{\mu}_3, \hat{\mu}_4) = \mu_7 - 3\mu_5\sigma^2 - 5\mu_3\mu_4 + 12\mu_3\sigma^4 \\ \Omega_{24} &= \Omega_{42} = Cov(\hat{\mu}_3, \hat{\mu}_6) = \mu_9 - 3\mu_7\sigma^2 - \mu_3\mu_6 - 6\mu_4\mu_5 + 18\mu_5\sigma^4 \\ \Omega_{34} &= \Omega_{43} = Cov(\hat{\mu}_4, \hat{\mu}_6) = \mu_{10} - 4\mu_3\mu_7 - \mu_4\mu_6 - 6\mu_5^2 + 24\mu_3\mu_5\sigma^2\end{aligned}$$

We define a function $g(\mu_2, \mu_3, \mu_4, \mu_6) = (\mu_6 - 15\mu_2\mu_4 - 10\mu_3^2 + 30\mu_2^3)/\mu_2$, then we have the gradient matrix:

$$\mathbf{D} = \left[\frac{\partial g}{\partial \mu_j} \right] \Big|_{1 \times 4} = \left\{ \left(-\frac{\mu_6}{\mu_2^2} + 10\frac{\mu_3^2}{\mu_2^2} + 60\mu_2 \right), -20\frac{\mu_3}{\mu_2}, -15, \frac{1}{\mu_2} \right\}$$

Following the delta theorem, we can obtain the limit distribution of the sample cumulant ratio \hat{C}_6/\hat{C}_2 .

$$\frac{\hat{C}_6}{\hat{C}_2} \xrightarrow{d} N\left(\frac{C_6}{C_2}, \frac{\mathbf{D}\mathbf{\Omega}\mathbf{D}'}{n}\right)$$

Then, the variance of the sample statistic:

$$\begin{aligned} Var\left(\frac{\hat{C}_6}{\hat{C}_2}\right) = \frac{\mathbf{D}\Omega\mathbf{D}'}{n} = & [10575 - 30m_{10} + m_{12} + 18300m_3^2 + 2600m_3^4 - 225(-3 + m_4)^2 - 7440m_3m_5 \\ & - 520m_3^3m_5 + 216m_5^2 - 2160m_6 - 200m_3^2m_6 + 52m_3m_5m_6 + 33m_6^2 \\ & + (-3 + m_4)(10(405 - 390m_3^2 + 10m_3^4 + 24m_3m_5) - 20(6 + m_3^2)m_6 + m_6^2) \\ & + 840m_3m_7 - 12m_5m_7 + 345m_8 + 20m_3^2m_8 - 2m_6m_8 - 40m_3m_9]\sigma^8/n \end{aligned}$$

, where the normalized central moments $m_r = \mu_r/\sigma^r$. For normal distribution, we have:

$$Var\left(\frac{\hat{C}_6}{\hat{C}_2}\right) = \frac{720\sigma^8}{n}$$

With similar procedure, we can obtain the variance of eighth to second order cumulant ratio:

$$\begin{aligned} Var\left(\frac{\hat{C}_8}{\hat{C}_2}\right) = & [198450 + 1204m_{12} - 56m_{14} + m_{16} + 5376m_{11}m_3 - 112m_{13}m_3 - 9878400m_3^2 \\ & - 1254400m_3^4 + 46550(-3 + m_4)^4 + 1225(-3 + m_4)^5 - 112m_{11}m_5 - 1270080m_3m_5 \\ & - 250880m_3^3m_5 + 176400m_5^2 + 169344m_3^2m_5^2 - 6272m_3m_5^3 - 114660m_6 + 1693440m_3^2m_6 \\ & - 142688m_3m_5m_6 + 3136m_5^2m_6 - 784m_6^2 + 698880m_3m_7 - 48384m_5m_7 - 7168m_3^2m_5m_7 \\ & + 896m_3m_6m_7 + 512m_7^2 + 63630m_8 - 118720m_3^2m_8 - 112m_3m_5m_8 + 112m_5^2m_8 - 420m_6m_8 \\ & + 128m_3m_7m_8 + 59m_8^2 - 2m_{10}(-7(-885 + 224m_3^2 + 8m_3m_5) + m_8) \\ & - 70(-3 + m_4)^3(-7(1125 + 80m_3^2 + 8m_3m_5) + 70m_6 + m_8) \\ & + 70(-3 + m_4)^2(5040 + m_{10} + 30240m_3^2 + 560m_3m_5 - 56m_5^2 - 1078m_6 - 64m_3m_7 + 21m_8) \\ & - 92960m_3m_9 + 3808m_5m_9 - 16m_7m_9 + (-3 + m_4)(-1488375 + 5180m_{10} - 140m_{12} \\ & - 13524000m_3^2 + 4104240m_3m_5 + 62720m_3^3m_5 - 155232m_5^2 + 3136m_3^2m_5^2 \\ & + 343980m_6 - 62720m_3^2m_6 - 7840m_3m_5m_6 - 295680m_3m_7 + 9856m_5m_7 - 67830m_8 \\ & - 1120m_3^2m_8 - 112m_3m_5m_8 + 140m_6m_8 + m_8^2 + 8960m_3m_9)]\sigma^{12}/n \end{aligned}$$

For normal distribution, we have:

$$Var\left(\frac{\hat{C}_8}{\hat{C}_2}\right) = \frac{40320\sigma^{12}}{n}$$

4. Monto Carlo Simulation

To check whether our results for the errors of the various moments is reasonable or not, we have done Monte Carlo simulations. Experimentally, we calculated the various moments from measured event-by-event net-proton or net-charge multiplicity distributions. For simplify, we assume the particle and anti-particle are independently distributed as Poissonian distribution, which is an appropriate approximation. Then, the difference of two independent Poisson distributions distributed as "Skellam" distribution. Its probability density distribution is

$$f(k; \mu_1, \mu_2) = e^{-(\mu_1 + \mu_2)} \left(\frac{\mu_1}{\mu_2}\right)^{k/2} I_{|k|}(2\sqrt{\mu_1\mu_2})$$

, where the μ_1 and μ_2 are the mean value of two Poisson distributions, respectively, the $I_k(z)$ is the modified bessel function of the first kind. Then, we can calculate various moments

(M, σ, S, κ) and moment products $(\kappa\sigma^2, S\sigma)$ products of the Skellam distribution. The results are shown below:

$$\begin{aligned}
M &= \mu_1 - \mu_2 \\
\sigma &= \sqrt{\mu_1 + \mu_2} \\
S &= \frac{\mu_1 - \mu_2}{(\mu_1 + \mu_2)^{3/2}} \\
\kappa &= \frac{1}{\mu_1 + \mu_2} \\
S\sigma &= \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \\
\kappa\sigma^2 &= \frac{C_6}{C_2} = \frac{C_8}{C_2} = 1
\end{aligned}$$

To do the simulation, we set the two mean values of the Skellam distributions as $\mu_1 = 4.11, \mu_2 = 2.99$, which are similar with the mean proton and anti-proton number in most central Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV measured by STAR experiment. Then, we can generate random numbers as per the "Skellam" distribution. Fig. 1 shows a distribution sampled from the "skellam" population with 30 million events.

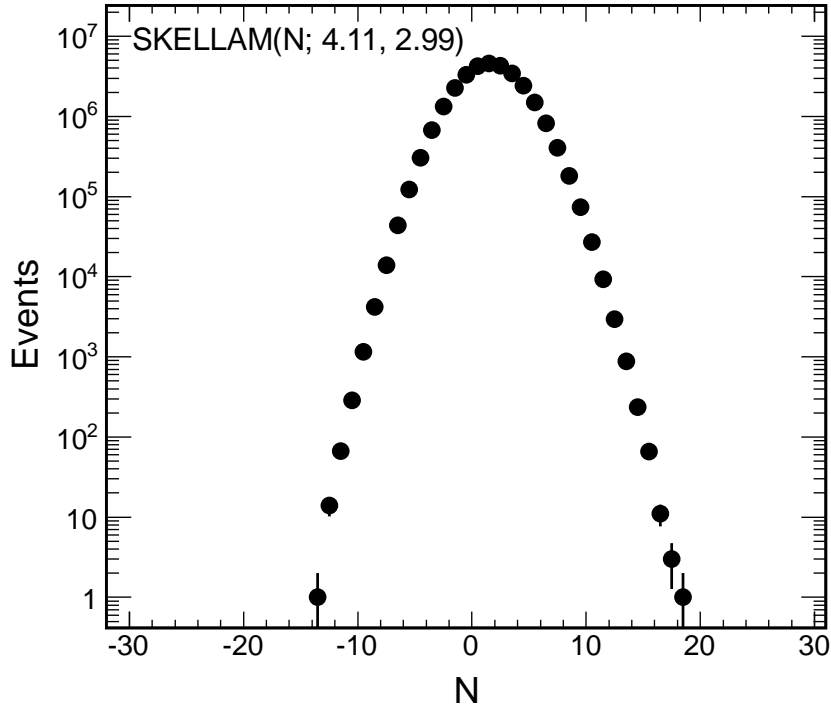


Figure 1. Sample distribution (30 million events) with the population distributed as skellam distribution.

In Fig. 2, we show the relative error as a function of events for various sample moments when the population is with "Skellam" distribution. We may find that the higher order moments are with larger relative errors, especially for the sixth and eighth order to second cumulant ratios.

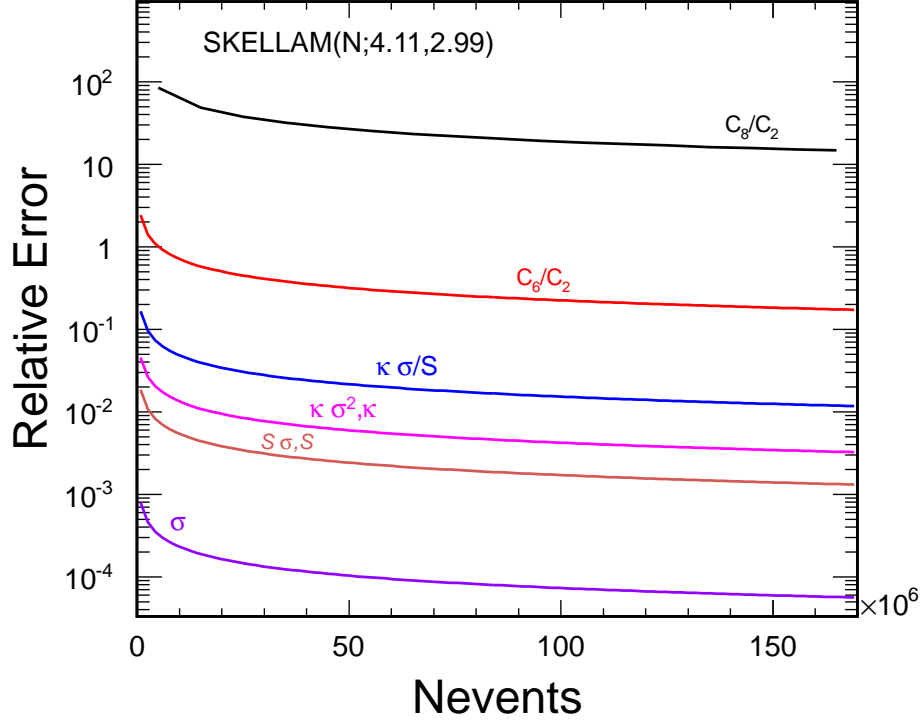


Figure 2. Relative error as a function of number of events for various sample moments calculated from the error formula shown in section 3 by assuming the population is with "Skellam" distribution.

As the input two parameters for "Skellam" distribution μ_1, μ_2 , are similar with the mean proton and anti-proton number for Au+Au 200 GeV data, we can estimate the number of events needed to achieve relative small errors for the moments studied at this energy. In year 2010 and 2011, STAR experiment has accumulated few hundreds million events of Au+Au collisions at $\sqrt{s_{NN}}=200$ GeV both for minbias and central trigger, from the Fig. 2, it allows us to study the higher moments of net-proton distributions up to sixth order with the acceptable errors.

According to the error formula shown in Section 3, for normal distributions, the errors for cumulants ratios are proportional to the standard deviation of the distribution as

$$\begin{aligned} error(\hat{S}\hat{\sigma}) &\propto \frac{\sigma}{\sqrt{n}} \\ error(\hat{\kappa}\hat{\sigma}^2) &\propto \frac{\sigma^2}{\sqrt{n}} \\ error(\frac{\hat{C}_6}{\hat{C}_2}) &\propto \frac{\sigma^4}{\sqrt{n}} \\ error(\frac{\hat{C}_8}{\hat{C}_2}) &\propto \frac{\sigma^6}{\sqrt{n}} \end{aligned}$$

Thus, with similar phase space coverage and number of events as high collision energy, we may get larger errors when we are doing moments analysis of net-proton distributions at low energy, as more nucleons are expected to be stopped in the central region due to nuclear stopping effect at low energy. While for the net-charge moment analysis, the case is opposite, as the total charged particle multiplicity is larger for high energy than that of low energy. On the other hand, with similar phase space coverage and number of events, the moments analysis of net-charge distributions should have larger errors than net-proton moments analysis at each collision

energy.

In the following, we will show various moments of 30 samples that independently and randomly generated from the "Skellam" distribution with different number of events (3, 30, 100 and 250 million) in Fig. 3 to 10. The two parameters for "Skellam" distribution is set to $\mu_1 = 4.11, \mu_2 = 2.99$. For comparison, we also put the expected value for each moment and the one standard deviation (σ) limit in those plots. For normal distribution, the probability for the value staying within $\pm 1\sigma$ around expectation is about 68.3%, that means in each panel of Fig. 3 to 10 about 20 out of 30 points should be in the range of $\pm 1\sigma$. From Fig.3 to 10, we may find that all of the moments are well satisfied this criteria and it indicates our error estimations for those moments are reasonable and can reflect the statistical properties of moments.

In Fig. 10, the eighth to second order cumulant ratio has large error bars even with 250 million events. Reliable results from this ratio need a large amount of statistics which may beyond the number of events we have accumulated.

5. Summary

Higher moments of conserved quantities have been extensively studied theoretically and experimentally. Due to the high sensitivity to the correlation length and direct connection to the thermodynamic susceptibilities, it can be used to probe the bulk properties, such as chiral phase transition, critical fluctuation at critical point, of hot dense matter created in the heavy ion collision experiment. To perform precise higher moments measurement, the error analysis is crucial for extracting physics message due to the statistic hungry properties of the moments analysis. We have estimated the errors for various moments that used in data analysis based on the Delta theorem in statistics and probability theory. Monto Carlo simulation is also done to check the validity of the error estimation. The simulation shows that the error estimations for various moments can reflect their statistical properties.

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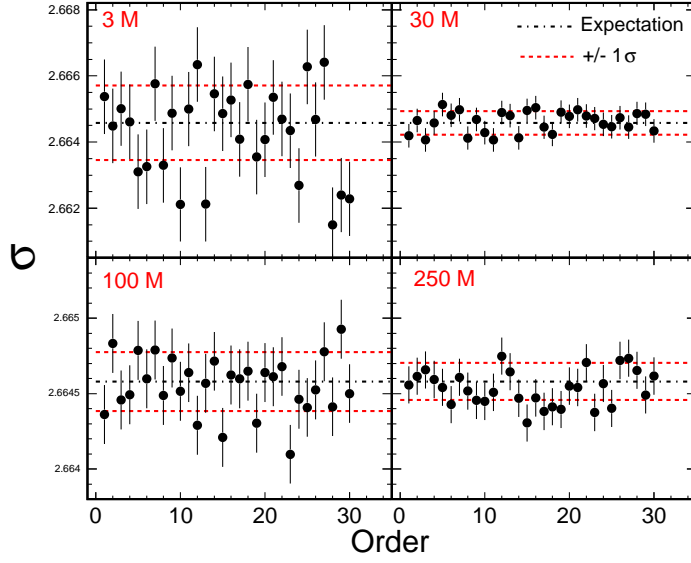


Figure 3. Standard deviation (σ) of 30 samples that independently and randomly generated from the Skellam distribution with different number of events (3, 30, 100, 250 million). The dashed lines are expectations and 1σ limits, respectively.

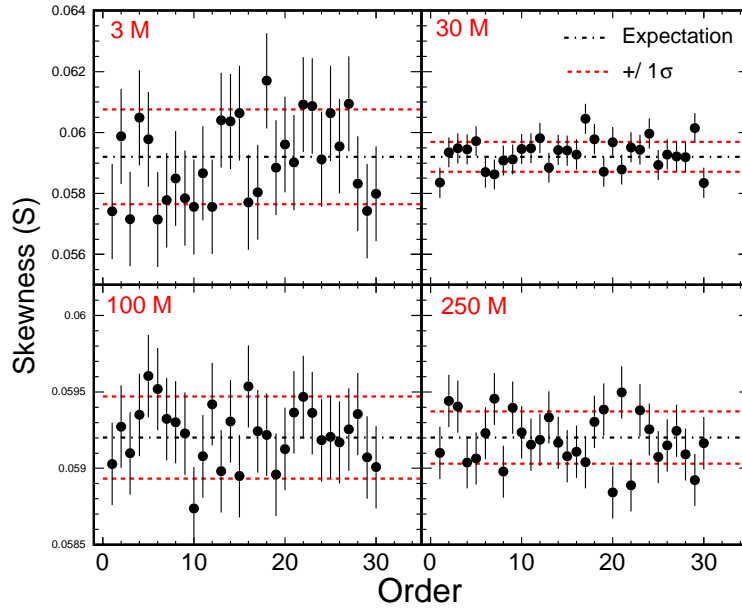


Figure 4. Skewness (S) of 30 samples that independently and randomly generated from the Skellam distribution with different number of events (3, 30, 100, 250 million). The dashed lines are expectations and 1σ limits, respectively.

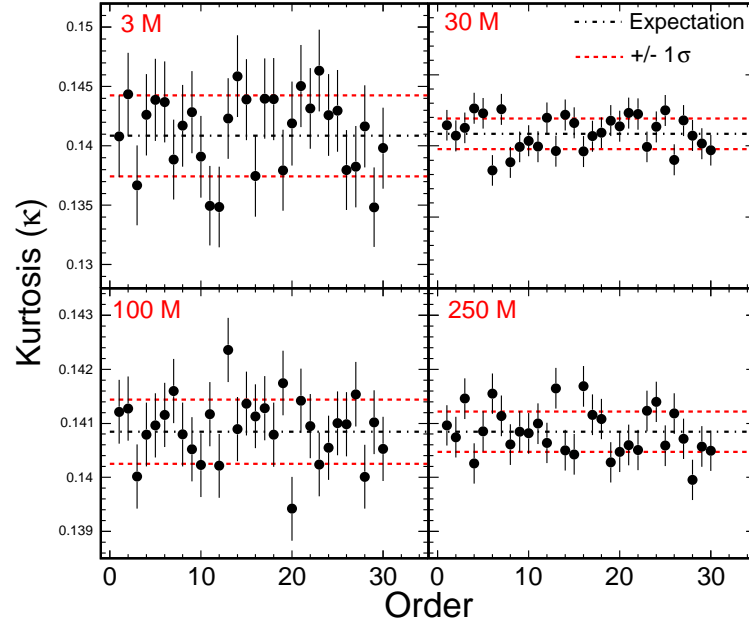


Figure 5. Kurtosis (κ) of 30 samples that independently and randomly generated from the Skellam distribution with different number of events (3, 30, 100, 250 million). The dashed lines are expectations and 1σ limits, respectively.

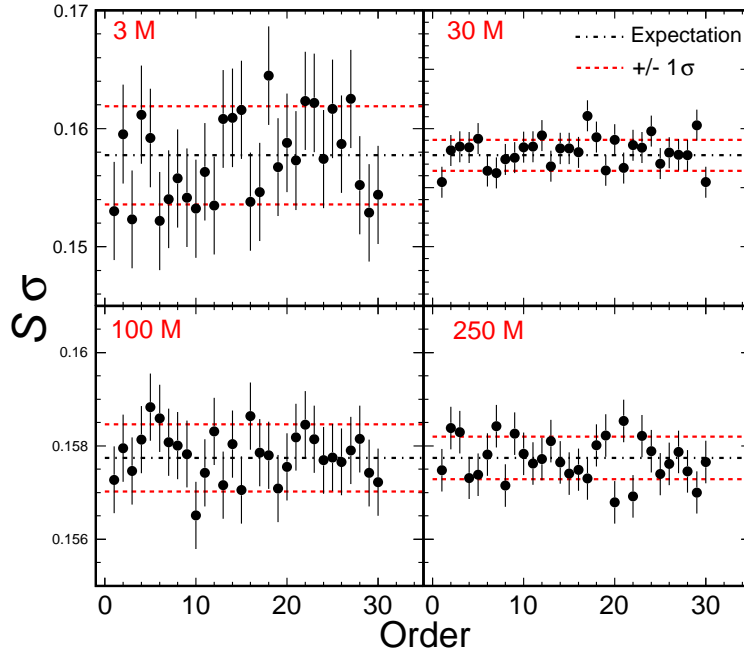


Figure 6. Skewness (S) times standard deviation (σ) of 30 samples that independently and randomly generated from the Skellam distribution with different number of events (3, 30, 100, 250 million). The dashed lines are expectations and 1σ limits, respectively.

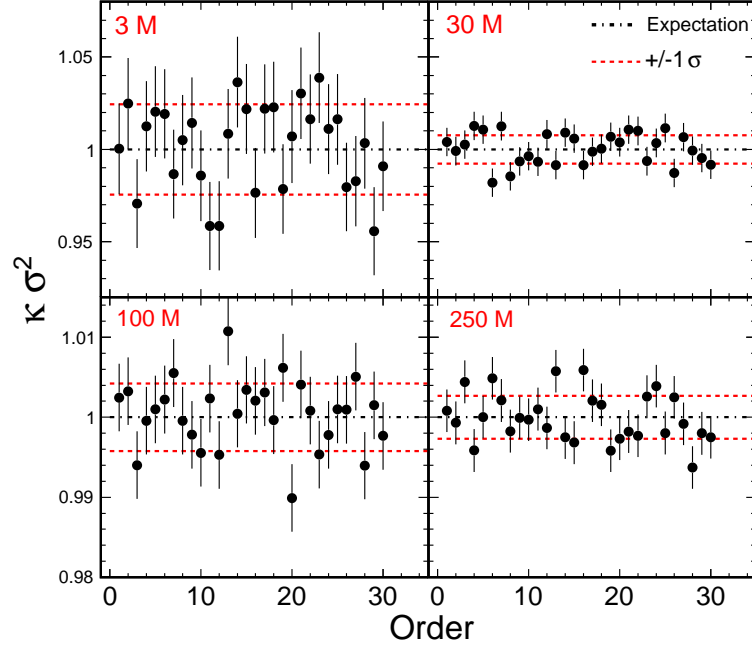


Figure 7. Kurtosis (κ) times variance (σ^2) of 30 samples that independently and randomly generated from the Skellam distribution with different number of events (3, 30, 100, 250 million). The dashed lines are expectations and 1σ limits, respectively.

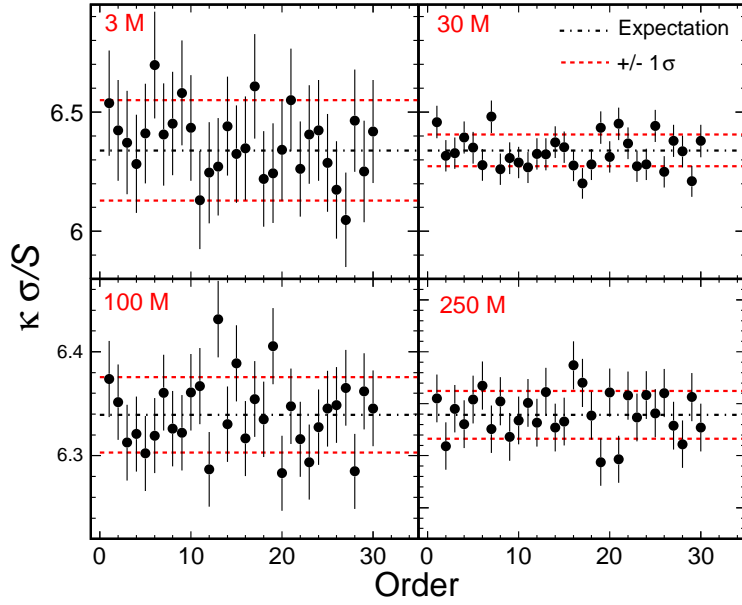


Figure 8. $\kappa \sigma / S$ of 30 samples that independently and randomly generated from the Skellam distribution with different number of events (3, 30, 100, 250 million). The dashed lines are expectations and 1σ limits, respectively.

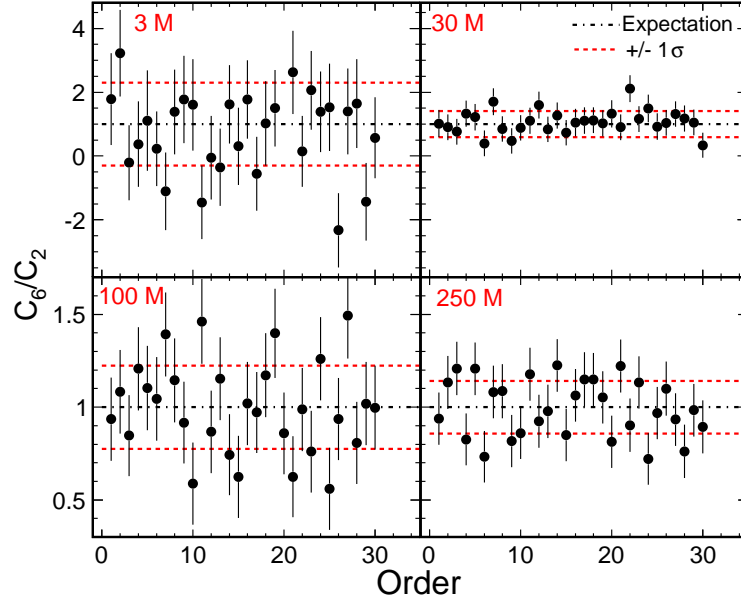


Figure 9. Sixth to second order cumulant ratio (C_6/C_2) of 30 samples that independently and randomly generated from the Skellam distribution with different number of events (3, 30, 100, 250 million). The dashed lines are expectations and 1σ limits, respectively.

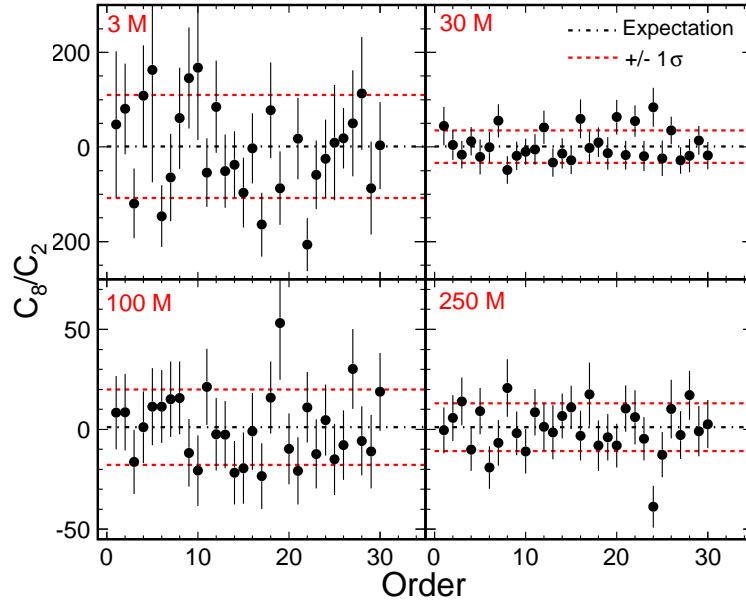


Figure 10. Eighth to second order cumulant ratio (C_8/C_2) of 30 samples that independently and randomly generated from the Skellam distribution with different number of events (3, 30, 100, 250 million). The dashed lines are expectations and 1σ limits, respectively.